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The problem of turbulence

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Abstract : We review the phenomenology of homogeneous, isotropic turbulence and explain why a proper theoretical understanding is still elusive.

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1. Introduction

Turbulent fluid flows are characterized by very strong transport properties. This is an everyday experience where one automatically stirs in the sugar in a cup of tea. Without the stirring the sugar would have dissolved by itself in the tea due to molecular diffusion. The time for this process can be estimated as L^2/D, where L is the characteristic dimension of the cup and D is the molecules diffusion coefficient. Estimating the typical dimension of a cup as 7 cms and the diffusion coefficient D as 10^{-3} cm^2/sec this process would take about 15 minutes. What the stirring does is, it produces a turbulent flow and the enhanced transport properties of a turbulent flow leads to a much larger effective value of D and that reduces the time of dissolution.

In general, turbulence occurs when the nonlinear terms in Navier Stokes equation for the fluid flow begin to dominate. Navier Stokes equation for fluids in the basic equation from which everything about flows has to follow and is written down as

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\frac{\nabla p}{\rho} + \upsilon \nabla^2 v + f \tag{1.1}$$

In the above, $\boldsymbol{v}$ is the velocity field, the left hand side is the total acceleration, the first term coming from an explicit variation and the second is associated with advection effects. The right hand side corresponds to the forces acting on the fluid. The first term is the force generated by a pressure gradient, the second corresponds to viscous dissipation and the last corresponds to all other forces that one may have *eg*. gravity, Coriolis (for a rotating system) *etc.* The viscosity that we show is the shear viscosity. By working with incompressible flows obeying

$$\nabla \cdot v = 0, \tag{1.2}$$

we avoid the occurrence of bulk viscosity on the right handside of eq. (1.1). The incompressibility condition makes the pressure field tagged to the velocity field for $f = 0$ or a solenoid f, *i.e.* when $\nabla \cdot f = 0$. In such cases, taking a divergence of eq. (1.1) leads to

$$\nabla^2 \frac{p}{r} = -\nabla \cdot (v \cdot \nabla) v . \tag{1.3}$$

This makes the pressure field somewhat akin to the nonlinear term and this shows up in a compact manner if we write eq. (1.1) in terms of the Fourier components $v(k, t)$ of the velocity field

$$v(r, t) = \frac{1}{(2\pi)^{D/2}} \int d^D r \, e^{i k.r} \, v(k, t) , \tag{1.4}$$

in the D-dimensional space that we are interested in. In terms of $v(k, t)$, eq. (1.1) for $f = 0$, becomes

$$\frac{\partial v_\alpha}{\partial t}(k,t) + vk^2 v_\alpha(k,t) = \sum_p M_{\alpha\beta\gamma}(k)\, v_\beta(p)\, v_\gamma(k-p) \tag{1.5}$$

and

$$M_{\alpha\beta\gamma}(k) = i\left[k_\beta P_{\alpha\gamma}(k) + k_\gamma \; P_{\alpha\beta}(k)\right] \tag{1.6}$$

with

$$P_{\alpha\beta}(k) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} . \tag{1.7}$$

If the force f satisfies $\nabla \cdot f = 0$, then

$$v_\alpha(k,t) + vk^2 v_\alpha = \sum_p M_{\alpha\beta\gamma}(k)\, v_\beta(p)\, v_\gamma(k-p) + f_\alpha . \tag{1.8}$$

To get a feel for when the nonlinear term in Novier Stokes equation dominates, we return to eq. (1.1) and note that (in the absence of external forces), the competition has to be between $(v \cdot \nabla)\, v$ and $v\, \nabla^2 v$. For a characteristic velocity V and a characteristic linear dimension L, the first term can be estimated as V^2/L and the second as V/L^2. The first dominates if

$$\mathrm{Re} = \frac{VL}{v} >> 1 . \tag{1.9}$$

The dimensionless number Re is called the Reynold's number. Turbulent flows generally occur at high Reynolds number.

As we noted before, all properties of flows must follow from eq. (1.1) with the appropriate boundary and initial conditions and as such it is a deterministic system. Naturally, the question is why should statistical mechanics play part in the description of turbulent flows. The answer lies in the mathematical characteristic of the turbulent flow. The flow is random, whose precise technical meaning is that the flow is sensitive to initial conditions. This implies that if one attempts to solve eq. (1.1) then for an infinitesimal change in initial conditions for the flow, the flows would be developing in very different manner after a finite time has elapsed. This, in turn, implies that talking about the velocity field $v(r, t)$ at any given time is not very useful. Instead, one needs to talk about an expectation value $\langle v(r, t)\rangle$, where the averaging is done over a vast number of realizations which differ from one another in the specification of the initial conditions. This is an ensemble average and assuming ergodicity holds, we can effectively get the same answer by averaging over time.

Is there a prior reason for us to believe that the flow depends strongly on initial conditions ? The answer to this question is not very clear but can be made quite plausible by considering a discrete version of eq (1.1) with $f = 0$. Working at a given point in space and replacing spatial derivatives by V_t / L where V_t is the velocity at time t and L is a characteristic length scale, the discrete version of eq (1.1) can be written as (the time derivative is $V_{t+1} - V_t$)

$$V_{t+1} = AV_t - BV_t^2 , \tag{1.10}$$

where A and B are constants. One of the constants can be absorbed in a scale for V and we can write eq. (1.10) in the form

$$V_{t+1} = r\,V_t(1 - V_t), \tag{1.11}$$

which is the well known logistic map for $0 < V_t < 1$ with the parameter r restricted to $0 < r < 4$ (the restriction on r ensures that the flow does not go out to infinity, rather remains bounded in the domain $0 < V_t < 1$). The sensitivity to initial conditions is most conveniently demonstrated for $r = 4$. In this case, the substitution $V_t = \sin^2 \theta_t$, leads to

$$\sin^2\theta_{t+1} = 4\sin^2\theta_t \cos^2\theta_t = \sin^2 2\theta_t \tag{1.12}$$

The resulting map on θ_t can be written as (if all angles are restricted to be between 0 and π).

$$\theta_{t+1} = 2\theta_t \quad \text{for } 0 \le \theta_t \le \frac{\pi}{2},$$

$$\theta_{t+1} = 2\left(\theta_t - \frac{\pi}{2}\right) \quad \text{for } \frac{\pi}{2} \le \theta_t \le \pi . \tag{1.13}$$

Writing $\frac{\theta_t}{\pi} = X_t$, we get

$$X_{t+1} = \begin{matrix} 2X_t & 0 \le X_t \le \frac{1}{2}, \\ 2X_t - 1 & \frac{1}{2} \le X_t \le 1. \end{matrix} \tag{1.14}$$

This map is known as Bernoulli shift and is very easily seen to be sensitive to initial conditions. To show this, we note that any number X_t between 0 and 1 can be written in the binary representation as

$$X_t = \frac{a_1}{2^1} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \ldots + \frac{a_n}{2^n} + \ldots, \tag{1.15}$$

where $a_n = 0$ or 1. For $a_1 = 0$, we get a number $X_t < \frac{1}{2}$ and for $a_1 = 1$, we get $X_t \ge \frac{1}{2}$. It is convenient to represent X_t by the string $(a_1 a_2 a_3 \ldots\ldots\ldots a_n \ldots\ldots\ldots)$. The action of the map in eq (1.14) is to shift the sequence one step to the left. To see this, suppose $a_1 = 0$, then

$X_{r+1} = \frac{a_2}{2} + \frac{a_3}{2^2} + \frac{a_4}{2^3} + \dots \frac{a_n}{2^{n-1}} + \dots$ and this is represented by $(a_2 a_3 \dots\dots a_n \dots\dots)$. If $a_1 = 1$ in X_r, then we have to subtract 1 after multiplying X_r by 2 (see eq. (1.14)) and that again leads to the string $(a_2 a_3 \dots\dots\dots a_n \dots\dots\dots\dots)$.

So the Bernoulli shift takes the string $(a_1 a_2 a_3 \dots\dots\dots a_n \dots\dots\dots\dots)$ and changes it to $(a_2 a_3 \dots\dots\dots a_n \dots\dots\dots\dots)$. Let us now start with two initial conditions which differ by 2^{-N}. The two initial conditions X_0 and X'_0, are then given by

$$X_0 = (a_1 a_2 a_3 \dots\dots\dots a_N \dots\dots\dots\dots),$$

$$X_0 = (a_1 a_2 a_3 \dots\dots\dots b_N \dots\dots\dots\dots).$$

After N steps of evolution according to the rules of eq. (1.14), we have

$$X_N = (\ a_N \ \dots\dots\dots\dots\dots),$$

$$X'_N = (\ b_N \ \dots\dots\dots\dots\dots).$$

The two numbers differ by 0(1). The separation in evolution occurs exponentially fast. Writing δX_0 as the initial separation and δX_N the separation after N steps,

$$\delta X_0 = \frac{1}{2N} \quad and \quad \delta X_N = \frac{1}{2}.$$

If we write for large N

$$\frac{\delta X_N}{\delta X_0} = e^{\lambda N}, \tag{1.16}$$

then for the above

$$\lambda = \ln 2. \tag{1.17}$$

The exponent λ defined in eq. (1.16) is called a Lyapunov and $\lambda > 0$ corresponds to sensitive dependence on initial conditions. As expected, for our example, λ is indeed positive as shown in eq. (1.17).

An evolution with positive Lyapunov exponent is called chaotic and what we have demonstrated above is that for non-negligible non-linearity, the solutions to Navier Stokes equation are probably chaotic and hence instead of a deterministic description, we seek a stochastic description.

We return again to eq. (1.1) and study the role of the nonlinear term on the energy balance. Multiplying scalarly with $\mathbf{v}$ and integrating over all space (let $f = 0$)

$$\frac{\partial}{\partial t}\int \frac{1}{2} v^2 d^D r = -\int v_\alpha v_\beta \frac{\partial v_\alpha}{\partial x_\beta} d^D r - \frac{1}{\rho}\int v_\alpha \frac{\partial p}{\partial x_\alpha} d^D r$$

$$+ \nu \int v_\alpha \nabla^2 v_\alpha \, d^D r \tag{1.18}$$

The left hand side is the time derivative of the total energy. For the first term on right hand side, we have

$$\int v_\alpha v_\beta \frac{\partial v_\alpha}{\partial x_\beta} d^D r = \int v_\beta \frac{\partial}{\partial x_\beta}\left(\frac{1}{2} v^2\right) d^D r$$

$$= \int \left[\frac{\partial}{\partial x_\beta} \left(\frac{1}{2} v^2 v_\beta \right) - \frac{1}{2} v^2 \frac{\partial v_\beta}{\partial x_\beta} \right] d^D r$$

$$= \int \frac{\partial}{\partial x_\beta} \left(\frac{1}{2} v^2 v_\beta \right) d^D r \, .$$

Gauss's theorem now converts the volume integral into a surface integral and with the usual boundary conditions that $v = 0$ on a rigid surface, we have the first term on the right hand side of eq. (1.18) vanishing. As for the second term an identical argument causes that to vanish and as for the third

$$\int v_\alpha \nabla^2 v_\alpha \, d^D r = -\int \frac{\partial v_\alpha}{\partial x_\beta} \frac{\partial v_\alpha}{\partial x_\beta} d^D r$$

and hence,

$$\frac{\partial E}{\partial t} = -\nu \int \frac{\partial v_\alpha}{\partial x_\beta} \frac{\partial v_\alpha}{\partial x_\beta} d^D r < 0 \, .$$

Consequently if external forces are absent, all motion must cease for $t \to \infty$. If we are to maintain the turbulence (as opposed to decaying turbulance) we must have an external force f acting on the system and supplying energy to it at the rate ε (energy per unit mass per unit time) such that

$$\varepsilon = \int f \cdot v \, d^D r = \nu \int \frac{\partial v_\alpha}{\partial x_\beta} \frac{\partial v_\alpha}{\partial x_\beta} d^D r \, . \tag{1.19}$$

The role of the nonlinear term is merely to transfer the energy from one length scale to another. What the nonlinear term does is to connect the velocity modes at momenta $\boldsymbol{k}, \boldsymbol{p}$ and $\boldsymbol{k} - \boldsymbol{p}$ and this connection helps transfer energy from one mode to another, keeping the total energy content of all modes fixed, *i.e.* $\sum_{\boldsymbol{k}} v_\alpha(\boldsymbol{k}) v_\alpha(-\boldsymbol{k})$ is unaltered by the nonlinear terms.

The turbulent velocity field being random, it pays to split the velocity field $\boldsymbol{v}(\boldsymbol{r}, t)$ into a mean part $\boldsymbol{v}_0$ and fluctuations $\boldsymbol{v}_f$ about the mean. We write

$$\boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{v}_0 + \boldsymbol{v}_f(\boldsymbol{r}, t) \tag{1.20}$$

and substitute in eq. (1.1) to obtain

$$\frac{\partial}{\partial t}(v_0 + v_f) + (v_0 \cdot \nabla) v_0 + (v_f \cdot \nabla) v_0 + (v_0 \cdot \nabla) v_f + (v_f \cdot \nabla) v_f$$

$$= -\frac{\nabla p}{\rho} + \nu \nabla^2 v_0 + \nu \nabla^2 v_f + F \tag{1.21}$$

Taking averages (ensemble or time) and using the fact that

$$\langle v \rangle = v_0 \tag{1.22}$$

(clearly $\langle v_f \rangle = 0$), we see

$$\frac{\partial v_0}{\partial t} + (v_0 \cdot \nabla) v_0 + \left\langle (v_f \cdot \nabla) v_f \right\rangle = -\left\langle \frac{\nabla p}{\rho} \right\rangle + \nu \nabla^2 v_0 + \quad . F \qquad (1.23)$$

The fluctuating field satisfies

$$\frac{\partial v_f}{\partial t} + (v_f \cdot \nabla) v_f = -\frac{\nabla p_f}{\rho} + \nu \nabla^2 v_f + N. \qquad (1.24)$$

where

$$N = -(v_f \cdot \nabla) v_0 - (v_0 \cdot \nabla) v_f - \left\langle (v_f \cdot \nabla) v_f \right\rangle. \qquad (1.25)$$

The force N represents the effect of the interaction between the mean flow and the random flow and is the source of energy that maintains the random flow. At this point, one takes a bold step to say that since v_f is random, the force N will be random and it is reasonable to study the fluctuations dynamics in a model where one replaces N by a solenoidal random force F and arrives at the so-called randomly stirred Navier Stokes equation

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\frac{\nabla p}{\rho} + \nu \nabla^2 v + F.$$

$$\nabla \cdot v = 0 \qquad (1.26)$$

or in momentum space

$$v_\alpha(k) + \nu k^2 v_\alpha(k) = \sum_p M_{\alpha\beta\gamma} v_\beta(p) v_\gamma(k-p) + F_\alpha(k)$$

where to specify the force F, we need to specify its correlation function. Keeping in mind the dimensional requirement that this force produces energy at the rates, and the requirement that $\nabla \cdot F = 0$, we can write the correlation in momemtum space as

$$\left\langle F_\alpha(k_1, t_1) F_\beta(k_2, t_2) \right\rangle = \frac{\varepsilon}{k_1^D} P_{\alpha\beta}(k_1)\, \delta(k_1 + k_2)\, \delta(t_1 - t_2),$$

where

$$P_{\alpha\beta}(k) = \delta_{\alpha\beta} \quad \frac{k_\alpha k_\beta}{k^2}$$

Using a tuning parameter y and working in frequency space,

$$\left\langle F_\alpha(k, \omega) F_\beta(k, \omega) \right\rangle = \frac{\varepsilon}{k^{D-4+y}} P_{\alpha\beta}(k)\, \delta(k + k')\, \delta(\omega + \omega'). \qquad (1.27)$$

The randomly stirred model is defined by eqs. (1.26) and (1.27) and its properties were first investigated by Forster *et. al* [11] and De Dominicis and Martin.[12]

2. Kolmogorov phenomenology

The Kolmogorov picture[13] (hereafter referred to as K 41) uses the concept of universality and dimensional analysis to make statements about the correlation functions of the fluctuating velocity field. To use universality, one needs to be in a parameter range where details are irrelevant. The first issue is, where is this parameter range. In the steady state picture that we are envisaging, the energy input by the random stirring force occurs at large length scales – length scales L which are of the order of the system size. The dissipation occurs over the very short length scales where the molecular dissipation term $\nu \nabla^2 v$ becomes dominant. In between, is a set of length scales where the mechanism of energy input or the mechanism of energy dissipation are not important. In this range, the nonlinear terms dominate and they simply transfer energy from one length scale to another at the constant rate ϵ, the rate at which it is injected to the system at large scales and taken away from it at short scales. Thus, there are three regions in momentum space.

i) region where $k \sim L^{-1}$, the small wave vector region-it is here that energy is injected into the system.

ii) region with $k \gtrsim k_d$, where k_d is a wave number formed from ε and the molecular viscosity ν. Clearly $k_d = (\varepsilon / \nu^3)^{\frac{1}{4}}$ and for $k \gtrsim k_d$, the molecular dissipative processes take over.

iii) region with $L^{-1} << k << k_d$, where the energy input mechanism and the dissipation mechanism do not play any role and universal results may be expected. This region is called the inertial range and here the correlation functions are determined by ε and k alone. The existence of an inertial range is dependent upon L^{-1} and k_d being very different from each other *i.e.* one requires $k_d L >> 1$. Noting that $\varepsilon \sim v^2/t \sim v^3/L$, we can write the conditions of existence of the inertial range as

$$1 << k_d L = (\mathrm{Re})^{3/4}.$$

Thus, a high Reynolds number guarantees the existence of a substantial inertial range.

One generally begins with the Kolmogorov 5/3 law for which one needs to introduce the energy spectrum. The total energy can be written as

$$\begin{aligned} E &= \frac{1}{2} \int \langle v_\alpha(k)\, v_\alpha(-k) \rangle \frac{d^D k}{(2\pi)^D} \\ &= \frac{1}{2} \int \langle v_\alpha(k, \omega)\, v_\alpha(-k, -\omega) \rangle \frac{d\omega}{2\pi} \cdot \frac{d^D k}{(2\pi)^D} \\ &= \frac{1}{2} \int \langle v_\alpha(r, t)\, v_\alpha(r, t) \rangle\, d^D r \\ &= \int E(k)\, dk. \end{aligned} \tag{2.1}$$

The quantity $E(k)$ in the last line of the above equalities is by definition the energy spectrum and in the inertial range is determined by ε and k alone. Dimensional analysis now yields

$$E(k) = C_k\, \varepsilon^{2/3} k^{-5/3} \tag{2.2}$$

the socalled 5/3 law, where C_k is a dimensionless number which is universal being independent of the nature of the fluid in which turbulence is occurring. From the list of identities in eq. (2.1), this leads to

$$C(k) = \langle v_\alpha(k)\, v_\alpha\,(-k)\rangle = k_0 \varepsilon^{2/3} k^{-\frac{2}{3}-D}\,, \tag{2.3}$$

where K_0 is a dimensionless constant related simply to C_k.

In coordinate space, we have

$$\langle v_\alpha(x)\, v_\alpha\,(x+r)\rangle = \int \left\langle v_\alpha(k) e^{ik.x} v_\alpha(k') e^{ik'.(x+r)} \right\rangle \times \frac{d^D k}{(2\pi)^D} \frac{d^D k'}{(2\pi)^D} d^D x$$

$$= \int \langle v_\alpha(k) v_\alpha(k')\rangle\, e^{ik.r}\, e^{i(k+k')\cdot x}\, d^D x\, \frac{d^D k}{(2\pi)^D} \frac{d^D k'}{(2\pi)^D}$$

$$= \int \langle v_\alpha(k) v_\alpha(-k)\rangle\, e^{ik.r}\, \frac{d^D k}{(2\pi)^D}$$

$$= k_0 \varepsilon^{2/3} \int \frac{e^{ik.r}}{k^{D+2/3}} \frac{d^D k}{(2\pi)^D}\,.$$

$$= k_0 \varepsilon^{2/3} \int \frac{e^{ik.r}-1}{k^{D+\frac{2}{3}}} \frac{d^D k}{(2\pi)^D} + \langle v_\alpha\,(x)^2\rangle$$

$$= -\tilde{k}_0\, \varepsilon^{2/3} r^{2/3} + \langle v_\alpha\,(\bar{x})^2\rangle,$$

leading to

$$\left\langle [v\,(x+r) - v(x)]^2 \right\rangle = \tilde{K}_0\, \varepsilon^{2/3}\, r^{2/3}\,. \tag{2.4}$$

For the higher order structure factors, dimensional analysis leads to

$$\left\langle |v\,(x+r) - v(x)|^p \right\rangle = C_p \varepsilon^{p/3} r^{p/3}\,. \tag{2.5}$$

The case of $p = 3$ is of special interest because in this case, one can actually prove that in the inertial range

$$\left\langle |v\,(x+r) - v(x)|^3 \right\rangle = -\frac{4}{5}\varepsilon\, r\,. \tag{2.6}$$

This is one of the very few exact results and hence we will give a detailed proof.

We begin with some properties of the two point functions

$$C_{ij}(r) = \left\langle \left[v_i\,(x+r) - v_i(x)\right]\left[v_j\,(x+r) - v_j(x)\right] \right\rangle\,. \tag{2.7}$$

Isotropy implies that C_{ij} cannot depend on any direction in space and consequently the tensor C_{ij} can be formed from the unit vector $\hat{n}$ in the direction of $\boldsymbol{r}$ Thus,

$$C_{ij}(r) = A(r)\,\delta_{ij} + B(r)\, n_i\, n_j\,. \tag{2.8}$$

Choosing the coordinate axis in the direction of r, calling the velocity component along r as v_r and the transverse ones v_t, we have

$$C_{rr} = A + B,$$
$$C_{tt} = A, \tag{2.9}$$
$$C_{rt} = 0\,.$$

Expanding the parenthesis in eq. (2.7),

$$C_{ij}(r) = 2\left\langle v_i\,(x)\, v_j\,(x)\right\rangle - 2\left\langle v_i\,(x+r) - v_j(x)\right\rangle . \tag{2.10}$$

Differentiating with respect to r,

$$\frac{\partial C_{ij}}{\partial r_i} = 0\,. \tag{2.11}$$

From eq. (2.8), this condition leads to

$$A'\,\frac{x_i}{r}\,\delta_{ij} + \frac{B'}{r}\,x_i\,n_i\,n_j + B\left[\frac{r_j}{r^2} - \frac{2r_i}{r^3}\,r_i\,\frac{r_i}{r}\right] = 0$$

or

$$A' + B' + \frac{2B}{r} = 0\,. \tag{2.12}$$

It follows that $r\,C'_{tt} = 2\,(C_{tt} - C_{rr})$ or

$$C_{tt} = \frac{1}{2r}\,\frac{d}{dr}\,(r^2\, C_{rr})\,. \tag{2.13}$$

In the inertial range $C_{ij} \propto r^{2/3}$ and hence

$$C_{tt} = \frac{4}{3}\,C_{rr}\,. \tag{2.14}$$

In the dissipative range, for very short distances r, the velocity difference is proportional to r and hence $C \propto r^2$

In that situation,

$$C_{tt} = 2\,C_{rr}\,. \tag{2.15}$$

Writing $B_{rr} = ar^2$, where a is a constant, we can use eqs. (2.10), (2.9) and (2.8) to write

$$\left\langle v_i(x+r)v_j\,(x)\right\rangle = \left\langle v_i(x)v_j\,(x)\right\rangle - ar^2\,\delta_{ij} + \frac{1}{2}\,ar^2 n_i n_j\,. \tag{2.16}$$

Differentiating the above relation

$$\left\langle \frac{\partial v_i}{\partial x_{1j}}(x_1)\,\frac{\partial v_j}{\partial x_{2j}}(x_2)\right\rangle = 15a, \quad \left\langle \frac{\partial v_i}{\partial x_i}(x_1)\,\frac{\partial v_i}{\partial x_{2i}}(x_2)\right\rangle = 0\,.$$

Since these relations hold for all differences $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$, it also holds for $\boldsymbol{r} \to 0$ and this

$$\left\langle \left(\frac{\partial v_i}{\partial x_j} \right)^2 \right\rangle = 15a , \quad \left\langle \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_l} \right\rangle = 0 . \tag{2.17}$$

For the mean energy dissipation, one has

$$\epsilon = \frac{1}{2} \nu \left\langle \left(\frac{\partial v_i}{\partial x_l} \frac{\partial v_e}{\partial x_i} \right)^2 \right\rangle = 15 a \nu \tag{2.18}$$

leading to $a = \epsilon / 15\nu$. In the dissipation range, consequently,

$$C_{ll} = \frac{2}{15} \varepsilon r^2 / \nu \quad C_{rr} = \frac{1}{15} \varepsilon r^2 / \nu \tag{2.19}$$

The three point velocity correlation is defined as

$$C_{ijk} = \left\langle \left(v_i (\boldsymbol{x}+\boldsymbol{r}) - v_i(\boldsymbol{x}) \right) \left(v_j (\boldsymbol{x}+\boldsymbol{r}) - v_j(\boldsymbol{x}) \right) \left(v_k (\boldsymbol{x}+\boldsymbol{r}) - v_k(\boldsymbol{x}) \right) \right\rangle \tag{2.20}$$

For a completely homogeneous and isotropic flow, the tensor $\langle v_i(\boldsymbol{x}) v_j(\boldsymbol{x}) v_k (\boldsymbol{x}+\boldsymbol{r})\rangle$ can only depend on δ_{ij} (the tensor is symmetric in i and j) and n_i. The most general form is

$$\langle v_i(\boldsymbol{x}) v_j(\boldsymbol{x}) v_k (\boldsymbol{x}+\boldsymbol{r})\rangle = C(r) \delta_{ij} n_k + D(r) (\delta_{ik} n_j + \delta_{jk} n_i) + F(r) n_i n_j n_k . \tag{2.21}$$

Differentiating with respect to r_k and using the equation of continuity

$$\frac{\partial}{\partial r_k} \left\langle v_i (\boldsymbol{x}) v_j (\boldsymbol{x}) v_k (\boldsymbol{x}+\boldsymbol{r}) \right\rangle = 0 . \tag{2.22}$$

Using eq. (2.21), steps similar to those leading to eq (2.13) give

$$\frac{\partial}{\partial r} \left[r^2 (3C + 2D + F) \right] = 0 ,$$

$$C' + 2(C+D)/r = 0 . \tag{2.23}$$

Integrating eq. (2.23), $3C + 2D + F = \text{constant}/r^2$, and requirement that C, D and F be finite at $r = 0$, leads to $3C + 2D + F = 0$ and hence

$$D = -(C + \tfrac{1}{2} rC') ,$$

$$F = rC' - C . \tag{2.24}$$

Expanding the parenthesis in eq. (2.20), we see that terms of the sort $\langle v_i(\boldsymbol{x}) v_j(\boldsymbol{x}) v_k(\boldsymbol{x})\rangle$ have to be zero since this tensor has to be constructed out of δ_{ij} alone and that is impossible. Tensors of the kind $\langle v_i(\boldsymbol{x}) v_j(\boldsymbol{x}) v_k(\boldsymbol{x}+\boldsymbol{r})\rangle$ and $\langle v_i(\boldsymbol{x}+\boldsymbol{r}) v_j(\boldsymbol{x}+\boldsymbol{r}) v_k(\boldsymbol{x})\rangle$ differ in sign but are equal in magnitude since n_i changes sign if $\boldsymbol{x}$ is interchanged with $\boldsymbol{x}+\boldsymbol{r}$. Thus,

$$C_{ijk} = 2\Big[\langle v_i(x)\, v_j(x)\, v_k(x+r)\rangle + \langle v_i(x)\, v_j(x+r)\, v_k(x)\rangle + \langle v_i(x+r)\, v_j(x)\, v_k(x)\rangle\Big] \tag{2.25}$$

Using eqs. (2.21) and (2.24)

$$C_{ijk} = 2\,(rC' + C)\,(\delta_{ij} n_k + \delta_{ik} n_j + \delta_{jk}\, n_i) + 6\,(rC' + C) n_i n_j n_k \tag{2.26}$$

With one of the coordinate axes along $\boldsymbol{n}$, the components of C_{ijk} follow as :

$$C_{rrr} = -12C,\ C_{rtt} = -2(C + rC'),\ C_{rrt} = \quad C_{ttt} = 0.$$

The relation among the non zero components is

$$C_{rtt} = \frac{1}{6}\frac{d}{dr}(r\, C_{rrr}\,). \tag{2.27}$$

Next, we find the relation between C_{ij} and C_{ijk}.

To do so, we make use of Navier Stokes equation to write

$$\frac{\partial}{\partial t}\Big\langle v_i(x)\, v_j(x+r)\Big\rangle = \Big\langle v_i(x)\, v_j(x+r\Big\rangle + \Big\langle v_i(x)\, v_j(x+r)\Big\rangle$$

$$= -\Big\langle v_l\,(x)\partial_l v_i(x)\, v_j(x+r)\Big\rangle - \Big\langle v_i\,(x) v_l(x+r)\partial_l v_j(x+r)\Big\rangle$$

$$+ \nu\Big\langle v_j(x+r)\nabla^2 v_i\,(x)\Big\rangle + \nu\Big\langle v_i(x)\nabla^2 v_j\,(x+r)\Big\rangle$$

$$-\Big\langle v_j(x+r)\partial_i\, \frac{P}{\rho}(x)\Big\rangle - \Big\langle v_i(x)\partial_j\, \frac{P(x+r)}{\rho}\Big\rangle \tag{2.28}$$

We note that $\langle v_j\,(x+r)\,\partial_i\,\frac{P(x)}{\rho}\rangle = \partial_i\,\langle v_j\,(x+r)\,\frac{P(x)}{\rho}\rangle$ and that the vector $v_j\,(x+r)\frac{P(x)}{\rho}$ is divergence free. There is no centrosymmetric divergence free vector which is finite at the origin and hence the fifth and sixth terms in the above equation vanish. The first and second terms are equal and so are the third and fourth, leading to

$$\frac{\partial}{\partial t}\Big\langle v_i\,(x)\, v_j\,(x+r)\Big\rangle = -2\partial_l\Big\langle v_l\,(x)\, v_i\,(x)\, v_j\,(x+r)\Big\rangle$$

$$+ 2\nu\,\nabla^2\Big\langle v_i\,(x)\, v_j\,(x+r)\Big\rangle. \tag{2.29}$$

Recalling $\Big\langle v_i\,(x)\, v_j\,(x+r)\Big\rangle = \Big\langle v_i\,(x)\, v_j\,(x)\Big\rangle - \frac{1}{2}\,C_{ij}$, we make use of the isotropy and homogeneity to write

$$\Big\langle v_i\,(x)\, v_j\,(x+r)\Big\rangle = \frac{1}{3}\,v^2\delta_{ij} - \frac{1}{2}\,C_{ij} \tag{2.30}$$

The three point function (eq. 2.20) can be written as

$$\Big\langle v_i\,(x)\, v_j\,(x)\, v_k(x+r)\Big\rangle = -\frac{1}{12}\,C_{rrr}\,\delta_{ij} n_k$$

$$+\frac{1}{12}\Big(\frac{1}{2}\,r\,C'_{rrr} + C_{rrr}\Big)\Big(\delta_{ik} n_j + \delta_{jk} n_i\Big) - \frac{1}{12}\big(r\,C'_{rrr} - C_{rrr}\big)\,n_i n_j n_k\,. \tag{2.31}$$

The time derivative of the kinetic energy $\frac{1}{2}v^2$ is the energy dissipation rate ε and thus $\frac{\partial}{\partial t}(\frac{1}{3}v^2) = -\frac{2}{3}\varepsilon$. A long but straight-forward calculation now casts eq. (2.29) in the form

$$-\frac{2}{3}\varepsilon - \frac{1}{2}\frac{\partial}{\partial t}C_{rr} = \frac{1}{6r^4}\frac{\partial}{\partial r}(r^4 C_{rrr}) - \frac{\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial C_{rr}}{\partial r}\right). \tag{2.32}$$

For very small r, we can put $r = 0$ on the left hand side and drop C_{rr} is comparison with ε. Multiplying through by r^4, integrating over r and using the fact that correlations vanish at $r = 0$,

$$C_{rrr} = -\frac{4}{5}\varepsilon r + 6\nu\frac{d}{dr}C_{rr}. \tag{2.33}$$

In the inertial range, the term involving ν can be dropped, and we obtain

$$C_{rrr} = -\frac{4}{5}\varepsilon r. \tag{2.34}$$

3. Beyond Kolmogorov

The Kolmogorov phenomenology discussed in the precceding section has an internal inconsistency which was pointed out by Landau. The rate at which energy traverses along the length scales was taken to be a constant. The rate of dissipation of energy at a given point x, should be the same ε independent of what it is a different point $x + r$. Now the energy dissipation rate at x is

$$\varepsilon(x) = \nu\int d^D r\left\langle\frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\right\rangle \tag{3.1}$$

over a small
ball around
the point x

and according to Kolmogorov's picture

$$K_{\varepsilon\varepsilon}(r) = \langle\varepsilon(x)\,\varepsilon(x+r)\rangle = C\,\varepsilon^2, \tag{3.2}$$

is independent or r. But ε is determined by a random variable and is likely to be governed by a distribution itself. This would imply that $K_{\varepsilon\varepsilon}(r)$ would be dependent on r. In that case

$$K_{\varepsilon\varepsilon}(r) = C\,\varepsilon^2\left(\frac{r}{L}\right)^{-\mu}, \tag{3.3}$$

where $\mu = 0$ in the Kolmogorov picture, but is different from zero if the Kolmogorov picture is violated. The measured dissipation rate (the measurement is one at a fixed point in space as a function of time and Taylor's frozen time scale hypothesis used to extract the spatial distribution) occassionally shows a sharp large fluctuations is called intermittency. With the existence of these fluctuations, the exponent μ is now no longer zero and for obvious reasons is called the intermittency exponent.

The immediate question is what happens to the p-th order correlations that we introduced in eq. 2.5, if we allow for fluctuations in the dissipation rate. The first answer was provided by Kolmogorov[15] and Obukhov[16] in 1962, when it was assumed that ε (a positive definite quantity) has a log normal distribution. A perturbative calculation of the width, σ, of the distribution (assuming a factoring approximation and Kolmogorov scaling for intermediate lines) suggests the form

$$\sigma^2 = A + 9\delta \ln \frac{L}{r}, \tag{3.4}$$

where δ is a universal number. If the mean of the distribution is m then

$$\left\langle \varepsilon^{p/3} \right\rangle = e^{pm/3} \; e^{p^2\sigma^2/18} \tag{3.5}$$

with the velocity Δv being dimensionally $(\gamma \varepsilon_r)^{1/3}$

we have

$$\left\langle |\Delta v|^p \right\rangle = \text{const.}\, r^{p/3} \left\langle \varepsilon_r^{p/3} \right\rangle$$

$$= \text{const.}\, r^{p/3} \; e^{pm/3} \; e^{p^2(A+9\delta \ln \frac{L}{r})/18} \tag{3.6}$$

The result for $p = 3$ is exact as we learnt in the previous section and that fixes m.

$$\left\langle |\Delta v|^3 \right\rangle = \text{const}\; \bar{\varepsilon}\, r$$

$$= \text{const}\; \langle \varepsilon_r \rangle\, r$$

$$= \text{const}\; e^m e^{\sigma^2/2}\, r,$$

leading to

$$e^{m + \frac{\sigma^2}{2}} = \bar{\varepsilon}. \tag{3.7}$$

Returning to eq. (3.6) and using eq. (3.7),

$$\left\langle |\Delta v|^p \right\rangle = C_0 \; \bar{\varepsilon}^{p/3} \; \bar{e}^{p\sigma^2/6} \; e^{\frac{p^2}{18}(A+9\delta \ln \frac{L}{r})} \, r^{p/3}$$

$$= C_p \; (\bar{\varepsilon})^{p/3} r^{p/3} \left(\frac{L}{r}\right)^{p(p-3)\delta/2}. \tag{3.8}$$

For $p = 6$,

$$\left\langle |\Delta v|^6 \right\rangle = C_6 \; \bar{\varepsilon}^2 \; r^2 \left(\frac{L}{r}\right)^{9\delta}.$$

Noting that $\varepsilon_r \sim |\Delta v|^3 / r$, we have

$$K_{\varepsilon\varepsilon} \propto \frac{1}{r^2} \left\langle |\Delta v|^6 \right\rangle \propto \left(\frac{L}{r}\right)^{9\delta},$$

leading to

$$\mu = 9\delta . \tag{3.9}$$

In the Kolmogorov Obukhov framework, once the intermittency exponent is known all the corrections to Kolmogorov scaling are known from eq. (3.8).

With increasing accuracy of experiments in determining the higher order structure factors it is possible to say with some confidence that eq. (3.8) is not borne out experimentally.

The basic feature of intermittency being the occurrence of rare events, it would be helpful if we look into the probability of rare events. To do so, we consider the binomial distribution – the problem of putting Q objects into two boxes and ask for the probability of P objects being in one box. The required probability $\rho(P|Q)$ is

$$\rho(P \mid Q) = \frac{Q!}{P!(Q-P)!} \frac{1}{2^Q} . \tag{3.10}$$

We will consider $Q >> 1$. On an average, half the objects will end up in box 1. If we ask for the probability of the number of objects P in box 1 being different from $Q/2$, then examination of eq (3.10) for $P \simeq Q/2$ gives,

$$\rho(P \mid Q) = \frac{1}{(2\pi Q)^{1/2}} e^{-\left(P-\frac{Q}{2}\right)^2/2Q} \tag{3.11}$$

the Gaussian distribution. For arbitrary P though, we can rewrite eq. (3.10), using Stirlings formula, as

$$\rho(P \mid Q) \simeq \frac{1}{2Q} \frac{Q^Q e^{-Q}}{(2\pi Q)^{1/2}} \frac{(2\pi P)^{1/2}}{P^P e^{-P}} \frac{[2\pi (Q-P)]^{1/2}}{(Q-P)^{Q-P} e^{-(Q-P)}}$$

$$= (2\pi Q)^{1/2} \left[\frac{P}{Q}\left(1-\frac{P}{Q}\right)\right]^{1/2} e^{-Q\ln 2}\, e^{-(Q-P)\ln\left(1-\frac{P}{Q}\right)} e^{-P\ln\frac{P}{Q}}$$

$$= (2\pi Q)^{1/2} [\alpha(1-\alpha)]^{1/2} , \tag{3.12}$$

where $\alpha = P/Q$ and

$$f(\alpha) = \ln 2 + (1-\alpha)\ln(1-\alpha) + \alpha \ln \bar{\alpha} \tag{3.13}$$

For $\alpha = 1/2$, $f(1/2) = 0$ and $f'(1/2) = 0$ and we have the Gaussian distribution that we anticipated in eq. (3.11).

But for $\alpha << 1/2$, we have

$$f(\alpha) = \ln 2 - \alpha + \alpha \ln \alpha , \tag{3.14}$$

which means that the probability distribution has the structure of an exponential distribution where large numbers (Q) appear multiplied by numbers of o(1). Writing $L = e^Q$ as a large number (or parameter)

$$\ln \rho \simeq - f(\alpha) \ln L, \tag{3.15}$$

which provides a different way of looking at the problem. The quantity $f(\alpha)$ is now a scaling index, with the difference that it is not a constant but a continuously varying function of α. Thus, this is a problem characterized by an infinite number of critical indices and hence this description is usually referred to as multiscaling or multifractal.

We can generalize this approach further by considering a conditional probability $\rho(X, L)$ where X and L are both large numbers. In a simple scaling approach, $\rho \sim L^{-\nu}$ and $X \sim L^{\mu}$, so that

$$\rho(X, L) = L^{-\nu}\, g(X / L^{\mu}) \tag{3.16}$$

(*e.g.* standard critical phenomena, where we consider magnetization m as a function of reduced temperature t and applied field h.

The scaling form is $m(t, h) = t^{\beta} f(h / t^{\beta\delta})$. If there are many critical indices labelled by subscripts i, then eq. (3.16) generalizes to

$$\rho(X, L) = \sum_i L^{\nu_i}\, g_i\left(\frac{X}{L^{\mu_i}}\right). \tag{3.17}$$

If we label ν_i and μ_i such that they both increase with i and assume

$$g_i(x) \sim \begin{cases} 1 & \text{for } x << 1 \\ 1 & \text{for } x \simeq 1 \\ 0 & \text{for } x >> 1 \end{cases}$$

then for $X \sim L^{\mu_i}$, $\rho(X, L) \sim L^{-\nu_i}$.

If instead of a discrete sum, we have a continuous distribution of indices

$$\rho(X, L) = \int d_{\alpha}\, L^{-f(\alpha)}\, g_{\alpha}\left(\frac{X}{L^{\alpha}}\right) \tag{3.18}$$

Now, if $X \sim L^{\beta}$ and $g_{\alpha}(x) \sim 0$ for $x >> 1$ and constant otherwise, we have

$$\rho(X, L) \sim L^{-f(\beta)} \text{ with } \beta = \ln x / \ln L, \tag{3.19}$$

One now has a recipe for a system suspected to exhibit multiscaling. Consider the system characterized by some large number (*e.g.* Reynolds number Re) and identify this parameter as L. One now measures quantity over a large range X and plots

$$\ln \rho / \ln L \quad \text{vs} \quad \ln X / \ln L.$$

If the description provided by eq. (3.19) is the correct one, then for different values of L, the date collapses on a single curve $f(\beta)$ which is the multifractal spectrum of critical indices. Realizing that the underlying set on which the dissipation is occurring is a multifractal,

Meneveau and Sreenivasan[17] came up with a physical picture of the energy transmission process which produces a multifractal distribution. This is pictured in Figure 1.

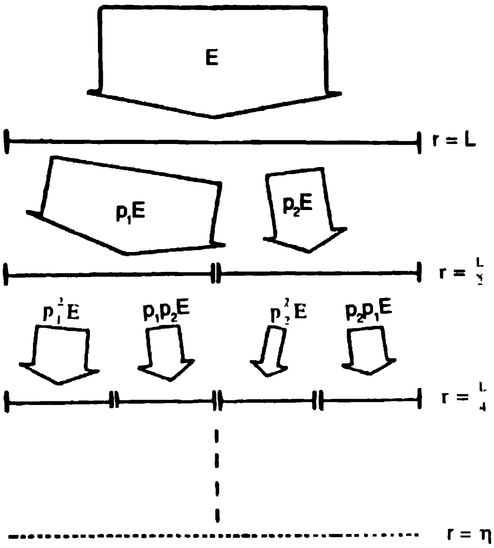

Figure 1. One-dimensional version of a cascade model of eddies, each breaking down into two new ones. The flux of kinetic energy to smaller scales is divided into nonequal fractions p_1 and p_2. This cascade terminates when the eddies are of the size of the Kolmogorov scale, η

We have taken a one-dimensional slice of the process and a large (scale L) energy containing eddy is shown dividing its energy into two eddies (scale $L/2$). The division however *does not occur equally* – the left hand eddy receives a fraction p_1 while the right hand side receives a fraction p_2 with $p_1 + p_2 = 1$.

To determine the multifractal characteristics of this process, we need to compute the quantity $\sum E_r^q$ where the sum is over the different eddies of size r at a scale $r = L/2''$ and q is an arbitrary integer. We expect

$$\sum E_r^q = E^q \left(\frac{r}{L}\right)^{(q-1)D_q} . \tag{3.20}$$

The set is a standard fractal if D_q is independent of q and multifractal if D_q is a nontrivial function of q. From the figure, it is clear that

$$\sum E_r^q = \sum_m E^q \, {}^nC_m \left(p_1^q\right)^m \left(p_2^q\right)^{n-m}$$

$$= E^q \left(p_1^q + p_2^q\right)^n \tag{3.21}$$

From eq. (3.20), $\sum E_r^q = E^q \left(\frac{1}{2n}\right)^{(q-1)D_q}$

leading to

$$D_q = -\frac{1}{q-1}\log_2\left(p_1^q + p_2^q\right). \tag{3.22}$$

With P_1 chosen to fit one of the moments, the indices D_q are known for all the moments and it was found by Menevau and Sreenivasan [17] that $p_1 = 0.7$ accounts for the data on all the different correlation functions.

A formula for the scaling behaviour of the n point correlation function was found by She and Leveque[18] based on heuristic arguments depending on a picture of the coherent structures in the problem. The velocity correlation can be written as

$$\left\langle |v(x+r) - v(x)|^n \right\rangle \propto \varepsilon^n\, r^{n/3}\, r^{\xi_n}, \tag{3.23}$$

here the anomalous dimension exponents ξ_n are given by

$$\xi_n = -\frac{2}{9}n + 2\left[1-\left(\frac{2}{3}\right)^n\right] \tag{3.24}$$

and are seen to be universal numbers.

The above formula for ξ_n fits the experimental result very well and it is a theoretical challange to derive ξ_n from Navier-Stokes equation.

Another way of expressing the above universality is in terms of Kolmogorov's refined similarity hypothesis.[19] One imagines dividing the spatial domain into a collection of ensembles each of them characterized by a fixed value of the locally averaged energy dissipation rate ε_r, where ε_r is the average of ε over a volume of linear dimension r. The refined hypothesis are

i) Over a range of scales r such that $r << L$, the probability density of the stochastic variable

$$V = \frac{v(x+r) - v(x)}{(r\,\varepsilon_r)^{1/3}} = \frac{\Delta v_r}{(r\,\varepsilon_r)^{1/3}} \tag{3.25}$$

depends only on the local Reynold's number $\mathrm{Re}_r = r(r\varepsilon_r)^{1/3} / \nu$

ii) If $\mathrm{Re}_r >> 1$, the probability density function of V does not depend on Re_r, either and is therefore universal.

It was found by Stolovitzy *et al.* [20] that the probability density function of V also depends or r when r is small. The remarkable thing about V is that it contain two different types of quantities $-\Delta v_r$ which is an inertial range quantity and ε_r which is a mixed quantity since it is the dissipation rate averaged over an inertial range scale. The inertial range and dissipation range cover widely different scales and yet the probability distribution of V is universal.

We end this section with a discussion of extended self similarity (ESS). [21,22]

If the n-th order structure factor (eq. (2.5) or (3.23)) is plotted against r, on a log-log plot the linear behaviour which is expected for small r shows deviations when r approaches the dissipation length scale. In fact, the linearity is at best seen over a couple of decades. Now using eq (2.6) one notes that the third order structure factor is proportional to r and if the available experimental or numerical data is plotted exhibiting the n-th order structure factor S_n against S_3 on a log-log plot the linearity persists over a longer range. This is called extended self similarity. In fact, one can plot $\log S_n$ vs $\log S_m$ and can get the linearity to persist into the dissipations range. This phenomenon is referred to as generalized extended self similarity. This is not entirely unexpected. Deviations from perfect scaling occur for both S_n and S_m and the imperfections "cancel" in a S_n *vs.* S_m curve.

4. Some theory

We now begin by demonstrating the main difficulty behind setting up a theory for turbulence. We return to eq. (1.26) with the definitions of M and P given by eqs. (1.6) and (1.7). The two quantities which we focus on are the response functions $G(\kappa, \omega)$ and the correlations function $C(k, \omega)$ defined as

$$P_{\alpha\beta} G(k,\omega) = \left\langle \frac{\partial v_\alpha(k,\omega)}{\partial f_\beta(k',\omega')} \right\rangle \frac{1}{\delta(k+k')\,\delta(\omega+\omega')} \tag{4.1}$$

and

$$P_{\alpha\beta} C(k,\omega) = \left\langle v_\alpha(k,\omega)\, v_\beta(k',\omega') \right\rangle \frac{1}{\delta(k+k')\,\delta(\omega+\omega')} \tag{4.2}$$

In the absence of the nonlinearity in E, (1.8), the response function is $G_0(k, \omega)$ given by

$$G_0^{-1}(k,\omega) = -i\omega + \nu k^2 \tag{4.3}$$

The nonlinearity changes G_0 to the full G by Dysons' equation which reads

$$G^{-1}(k,\omega) = -i\omega + \nu k^2 + \Sigma(k,\omega) \tag{4.4}$$

The physical interpretation of Σ is a relaxation rate. Treating the nonlinearity perturbatively, the one loop contribution $\Sigma^{(1)}(k,\omega)$ to Σ is given by

$$\Sigma^{(1)}(k,\omega) = \int \frac{d^D p}{(2\pi)^D} \int \frac{d\omega'}{2\pi} M_{\alpha\beta\gamma}(k)\, M_{\sigma\upsilon\alpha}(k)\, P_{\beta\sigma}(P)\, P_{\upsilon\gamma}(k-p)$$

$$\times\, G_0(p,\omega')\, C_0(k-p,\omega-\omega') \tag{4.5}$$

Where $C_0(k, \omega)$ is the zeroth order correlation function, which is clearly $|G(k,\omega)|^2 <ff>$.

We now introduce an approximation known as self consistent mode coupling and anticipate that in the Kolmogorov (scaling), limit, the contribution of the nonlinear terms to the self energy will dominate the molecular viscosity diffusion rate νk^2. Thus eq (4.4) becomes,

$$G^{-1} = -i\omega + \Sigma(k,\omega) \tag{4.6}$$

For the self consistent mode coupling approximation, one makes the assumption that replacing G_0 and C_0 on the right hand side of eq. (4.5) by the full response functions G and the full correlation function C leads to the full self energy Σ. Thus, the self-consistent mode coupling involves in the diagrammatic language a sum over a class of diagrams of all orders. The self energy is

$$\Sigma(k,\omega)=\int \frac{d^D p}{(2\pi)^D}\frac{d\omega'}{2\pi} M_{\alpha\beta\gamma}(k)\,P_{\beta\sigma}(P)\,P_{\upsilon\gamma}(k-p)\,M_{\sigma\upsilon\alpha}(p).$$

$$G(p,\omega')\,C(k-p,\omega-\omega') \tag{4.7}$$

The diagrammatic representations of this approximation is shown in Figure 2(a). There must be a similar approximation for the correlation function for eq (4.7) to be useful. This complication could be avoided for a system with fluctuation dissipation theorem (FDT) where the correlations and response functions are related. However, a FDT can be found only if the equation of motion leads to an equilibrium distribution function as $t \to \infty$. But there is no equilibrium distribution function associated with eq. (1.8) and hence we need on approximation for $C(\kappa,\omega)$ and in the above scheme this is shown in Figure (2b).

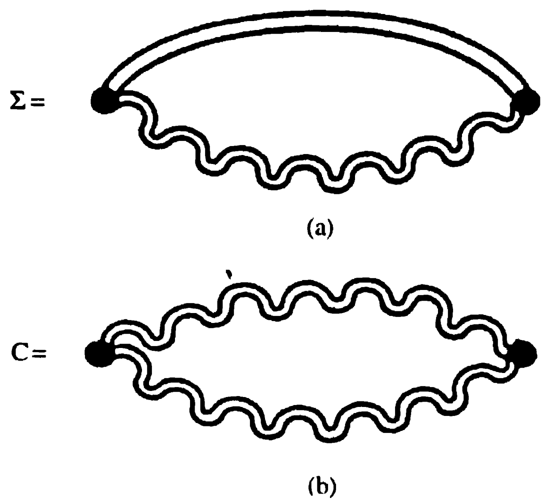

Figure 2 Diagrammatic representation of the self energy and correlation function.

Let us now examine the integrand I of eq. (4.7) when the momentum $\boldsymbol{P}$ becomes nearly the same as $\boldsymbol{k}$ or equivalently $\boldsymbol{q}=\boldsymbol{k}-\boldsymbol{p}\to 0$. In this limit the integrand for the zero frequency self energy (*i.e.* $\omega=0$) becomes

$$I=M_{\alpha\beta\gamma}(k)\,M_{\sigma\upsilon\alpha}(k)\,P_{\beta\sigma}(k)\,G(k,\omega')\,P_{\upsilon\gamma}(q)\,C(q,-\omega') \tag{4.8}$$

Since the momentum q is small *i.e.* $q<<k$, in the above integrand, the integration over ω', will sample mainly sample small frequencies *i.e.* frequencies of the order of q^z, where z is some as yet unknown dynamical exponent. This ensures that $G(k,\omega')$ reduces to $\Sigma(k)^{-1}$. Thus eq (4.7) reduces to

$$\Sigma(k)=M_{\alpha\beta\gamma}(k)\,M_{\sigma\upsilon\alpha}(k)\,P_{\beta\sigma}(k)\,/\,\Sigma(k)$$

$$\times\int\frac{d^D q}{(2\pi)^D}\,P_{\upsilon\gamma}(q)\,\frac{d\omega'}{2\pi}\,C(q,\omega')$$

$$\frac{M_{\alpha\beta\gamma}(k)\,M_{\sigma\upsilon\alpha}(k)}{\Sigma(k)}\,P_{\beta\sigma}(k)\int\frac{d^Dq}{(2\pi)^D}\,P_{\upsilon\gamma}(q)\,C(q),\tag{4.9}$$

where $C(q)$ is the equal time correlation function. According to the Kolmogorov phenomenology, expressed in eq (2.3) $C(q) \sim q^{-\left(\frac{2}{3}+D\right)}$ and using this in eq. (4.9), the self energy $\Sigma(k)$ follows from eq. (4.9) as

$$[\Sigma(k)]^2 \propto k^2 \int \frac{dq}{q^{1+\frac{2}{3}}},\tag{4.10}$$

an integral which diverges for $q \to 0$ and has to be cut off at some low momentum scale k_0. It follows that $\Sigma(k) \sim k$ instead of $k_{2/3}$ which would be consistent with Kolmogorov. Thus, the procedure outlined above for using Navier Stokes equation to arrive at the Kolmogorov spectrum, does not work.[21] It fails because the large effect in the dynamics of the Navier Stokes fluid is the advection of small eddies by the large ones and this has nothing to do with the Kolmogorov spectrum. It is only after this sweeping effect of the larger eddies has been explicitly removed that one can hope to obtain the Kolmogorov results. The important thing is to arrive at a strategy for removing the sweeping effect.

We can do so by looking at a high frequency dynamic screening effect.[22] The self energy of eq. (4.6) has the behaviour $\Sigma(k, 0) \sim k^{2/3}$ at low frequencies. At high frequencies this behaviour is suppressed and power counting in eq. (4.7) shows that

$$\sum(k, \omega >> \Sigma(\omega)) \sim k^2 / (-i\omega)^2\tag{4.11}$$

A one parameter scaling function for $\Sigma(k, \omega)$ is

$$\sum(k,\omega) = \Gamma_0\, k^{2/3}\left(1+\alpha\,\frac{-i\omega}{\Gamma_0\, k^{2/3}}\right)^{-2},\tag{4.12}$$

where α is a number of order unity.

The corresponding correlation function which is normalized to the correct low frequency behaviour is

$$C(k,\omega) = \frac{1}{k^{11/3}\,\Sigma(k)}\left[\frac{1}{1+\frac{-i\omega}{\Sigma(k,\omega)}} + c.c.\right]\tag{4.13}$$

The screening is evident in the frequency integration which would give zero since $\Sigma(k, \omega) \sim (-i\omega)^{-2}$ for high frequencies. Consequently the difficulty encountered in eq. (4.9) would not arise. The above form is valid for frequencies lower than a cut off frequency ω_0, above which the correlations function becomes a Lorentzian once again which restore the equal time form of the time dependent correlation function. This crossover from the form shown above to the Lorentzian is important for the static correlation function but can be ignored in the frequency integration of eq (4.7). The frequency integration in eq. (4.7) can be written as (for zero external frequency)

$$\int d\omega'\, G(p,\omega')\, C(q, -\omega')$$

$$= \int d\omega' \frac{[-i\omega' + \Sigma(p)]^{-1}}{q^{11/3}\,\Sigma(q)} \left\{ \frac{1}{1 + \frac{i\omega'}{\Sigma(q,-\omega')}} + c.c. \right\}$$

$$= \frac{1}{\Sigma(q)\,q^{11/3}} \cdot \frac{1}{1 + \frac{\Sigma(p)}{\Sigma(q,-i\,\Sigma(p))}}$$

$$= \frac{1}{q^{11/3}} \cdot \frac{1}{q^{2/3} + p^{2/3}}\, S\,, \tag{4.14}$$

where

$$S = \frac{q^{2/3} + p^{2/3}}{1 + \frac{p^{2/3}}{q^{2/3}} \left(1 + \alpha \frac{p^{2/3}}{q^{2/3}} \right)^2} \tag{4.15}$$

is the screening factor, which is unity when $\alpha = 0$, *i.e.* the frequency dependence of Σ is ignored. For $q = 0$, the a symptotic behaviour of S is $p^{2/3}\,\alpha^{-2}\,q^2 / p^2$ and prevents the integral in eq. (4.7) from diverging when $q \to 0$. In the above manipulations, we have maintained the pole approximations for the response function G since its role is secondary. We have essentially made the statement that in the time dependent correlation function $\langle v(k,t)\,v(k,t+\tau)\rangle$, the small τ behaviour is to be screened out in order to arrive at the Kolmogorov spectrum. This has to be done to remove the sweeping effect and unravel the behaviour which we are interested in.

A different point of view due to L'vov and Procaccia [23] is to work entirely in coordinate space in a quasi– Lagrangian approach. The short distance singularity that we discussed at the beginning of this section is taken care of by defining a ball of locality in which the correlation function is properly cut off to make the theory finite, *i.e.* to remove the sweeping contribution. We speculate that what the earlier method does in time, the approach of L'vov and Procaccia does in space and it should be possible to explore the connection between the two keeping in mind Taylors' frozen turbulence hypothesis.

Yet another way out of the difficulty is supposedly offered by the renormalization group.[24] By construction, this procedure integrates over the high wave number Fourier components to explore the effect of the nonlinear terms at long wavelengths. Since the integration is over the high momentum modes the infrared problem does not arise. However, the difficulty with the approach lies in the emergence of an infinite number of marginal operators.

We now turn to the role of the infrared divergence in a higher order correlation function. The correlation to focus on is $K_{\varepsilon\varepsilon}(r)$ (see eq. (3.2)) which naturally leads to the intermittency exponents μ. Noting that $\varepsilon \sim \frac{v^3}{r}$, we can obtain $K_{\varepsilon\varepsilon}$ from the properties of

$$K(r) = \frac{1}{r^2} \left\langle v^3(x)\, v^3(x+r) \right\rangle$$

$$= \frac{1}{r^2} \int e^{ik.r}\, d^D k\, K(k), \tag{4.16}$$

where

$$K(k) = \sum_{p_1, p_2, q_1, q_2} \langle v(p_1)\, v(p_2)\, v(k - p_1 - p_2)\, v(q_1)\, v(q_2)\, v(-k - q_1 - q_2) \rangle . \tag{4.17}$$

In a factoring approximation, this implies

$$K(k) = \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} C(p_1)\, C(p_2)\, C(k - p_1 - p_2)$$

$$= K_0^3\, \varepsilon^2 \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \cdot \frac{1}{p_1^{D+\frac{2}{3}}} \frac{1}{p_2^{D+\frac{2}{3}}} \frac{1}{|k - p_1 - p_2|^{D+\frac{2}{3}}} \tag{4.18}$$

The leading divergence of this integral is when p_1 and p_2 both tend to zero independently and from that region of the phase space of the integrand, the contribution is proportional to $k^{-\left(D+\frac{2}{3}\right)} k_D^{-4/3}$ where k_D is a low momentum out off on the divergent integrals in p_1 and p_2. Insertion of this leading behaviour in eq. (4.16) and comparison with eq. 3.2, leads to $\mu = \frac{4}{3}$ which is totally at variance with the observed μ which is close to 0.25. Thus the infrared divergence has to be subtracted again. The screening approximation which rendered the two point function finite has to be modified to work on the $K(r)$. Here an effective summation over parquet graphs has to be carried out to implement the screening. Thus a new procedure has to be devised for removing the divergence in $K(r)$. It will be seen that every new correlation function will require a new trick for yielding finite answers. This is what makes turbulence an as yet unsolved problem.

A slightly different approach has been introduced of late by Polyakov.[25] This is an attempt to directly construct the probability distribution for the velocity difference $\Delta v_r = v(x + r) - v(x)$. Noting, from the extensive numerical work of Cheklov and Yakhot[26] and Hayot and Jayaprakash[27] that the essence of turbulence is contained in the one dimensional Burgers Equation, with spatially correlated noise, Polyakov made use of operator product expansion to construct the probability distribution for this case. The distribution is non-Gaussian in the tail which indicates the existence of intermittency and multifractality.

We have not discussed at all the special case of turbulence in two dimensions where an extra conserved quantity (in the inviscid limit) exists. This is the enstrophy defined as $\int \omega^2 d^2 r$, where $\omega = \nabla \cdot v$. It is enstrophy which cascades from long length scales to short length scales, while energy cascades from short to long scales. Kolmogorov spectrum holds for the energy cascade, while the enstrophy spectrum, on dimensional analysis, turns out to be k^{-3}. More careful investigations show logarithmic corrections. Turbulence in presence of rotation is expected to show a crossover from three to two dimensional behaviour as the rotation rate increases and is an interesting area of study.

The behaviour of a passive scalar in a turbulent velocity field is another interesting issue. If the density of the scalar is c, then the governing equation for the evolution of the density is

$$\frac{\partial c}{\partial t} + (v \cdot \nabla) c = D \nabla^2 c ,$$

where v is the turbulent velocity field. The nonlinear term $(v \cdot \nabla) c$ causes the diffusion coefficient D to become scale dependent and self consistent perturbation theory shows that $D \sim L^{4/3}$, where L is the length scale at which the diffusion coefficient is being considered. This enhancement of D at large length scale is at the root of the effectiveness of turbulent diffusion – a fact which we mentioned at the very beginning of this article.

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